Tests of Uniform Convergence of Series

Th 1. Weierstrass's M-test – A series of function Σf_n will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series ΣM_n of positive numbers such that for all $x \in [a, b]$

$$|f_n(x)| \leq M_n \text{ for all } n$$

Let $\epsilon > 0$ be a positive number.
Since ΣM_n is convergent, therefore there exists a positive integer N such that

$$|M_{n+1} + M_{n+2} + \text{——} + M_{n+p}| < \epsilon$$

$$\forall\, n \geq N,\; p \geq 1 \qquad \text{(I)}$$

Hence for all $x \in [a, b]$ and for all $n \geq N$, $p \geq 1$, we have

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$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| \leq$$

$$|f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)| \quad — (2)$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon \quad — (3)$$

(2) and (3) imply that Σf_n is uniformly and absolutely convergent on $[a, b]$.

Th 2 **Abel's test** — If $b_n(x)$ is a positive monotonic decreasing function of n for each fixed value of x in the interval $[a, b]$ and $b_n(x)$ is bounded for all values of n and x concerned and if the series $\Sigma u_n(x)$ is uniformly convergent on $[a, b]$, then so also is the series $\Sigma b_n(x) u_n(x)$

Since $b_n(x)$ is bounded for all values of n and for x in $[a, b]$, therefore there exists a number $K > 0$ independent of x and n, such that for all $x \in [a, b]$

$0 \leq b_n(x) \leq K$ (for $n = 1, 2, 3, \ldots$) — ①

Again since $\sum U_n(x)$ converges uniformly on $[a, b]$, therefore for any $\epsilon > 0$, we can find an integer N such that

$$\left| \sum_{r=n+1}^{n+p} U_r(x) \right| < \frac{\epsilon}{K}, \forall \; n \geq N, \; p \geq 1 \quad \text{— ②}$$

Hence using Abel's lemma we get

$$\left| \sum_{r=n+1}^{n+p} b_r(x) U_r(x) \right| \leq b_{n+1}(x) \max_{q=1,2,\ldots p} \left| \sum_{r=n+1}^{n+q} U_r(x) \right| < K \cdot \frac{\epsilon}{K} = \epsilon$$

for $n \geq N$, $p \geq 1$, $0 \leq x \leq b$.

$\Rightarrow \sum b_n(x) U_n(x)$ is uniformly cgt on $[a, b]$